Scaling violations with textures in two-dimensional phase ordering

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Scaling violations are found in the phase-ordering two-dimensional Heisenberg [O(3)] model, which has nonsingular topological textures, under dissipative nonconserved dynamics. Three separate length scales are found: L_T characterizes the scale of individual textures, L_N characterizes the separation between textures, and L_C characterizes the distance between oppositely charged textures.

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Dynamical scaling has had great success in describing phase-ordering systems with purely dissipative local dynamics [1]. The scaling hypothesis is that the evolution of the system at late times can be characterized by a single growing length scale $L(t)$, which scales the two-point correlations of the system. As yet, however, there is no general approach to determine whether a particular system scales. Conserved spherical systems can be shown to violate scaling [2], while most other vector systems that have been investigated seem to scale fairly well. Exceptions are provided by one- and two-dimensional systems with weakly interacting nonsingular topological textures. Textures have both an internal scale characterizing the size of an individual texture, and external scales characterizing the texture separation. If these scales evolve with different growth laws then scaling will be violated.

The simplest system with topological textures is the one-dimensional (1D) XY [O(2)] model, in which the textures are windings and antiwindings of the XY phase along the system. These and all other textures are nonsingular, without the singular core structure of, e.g., vortices or domain walls. In 1D, anomalous growth laws for both nonconserved and conserved dynamics [3] indicate scaling violations [4] that have recently been observed and explained [5]. In this paper, we explore two-dimensional (2D) systems with textures, which we find also violate dynamical scaling. Unlike the 1D systems, however, no extended scaling description is found to apply to the two-point correlations.

2D Heisenberg [O(3)] systems support topological textures—also variously called skyrmions, instantons, or "baby skyrmions." An isolated texture can be pictured as a stereographic projection of an order-parameter sphere onto the plane of the system [4]. The intrinsic scale of the texture is then proportional to the radius of the projected sphere. These 2D textures have recently been of particular interest: e.g., in cosmological skyrmionic strings [6], in quantum antiferromagnets [7], and also in particle physics [8]. In all of these cases, 2D textures provide weakly interacting, localized, but nonsingular, excitations.

Textures have an associated topological charge which is quantized. After an arbitrary sign choice, isolated textures have total charge $+1$, while isolated antitextures have total charge -1. Static solutions consisting solely of texture (or solely of antitexture) configurations, in a (hard-spin) nonlinear σ model, were discussed by Belavin and Polyakov [9]. The solutions are notable because the energy of the system is independent of the overall scale and also of the locations of the individual textures, each one of which contributes unit charge and 8π energy [using Eq. (2), below]. These quasistable minimal energy solutions demonstrate the weak interactions of 2D textures. On the other hand, systems with both textures and antitextures are never static [9]. However, the dynamics and evolution of generic textured systems is relatively unexplored [8,10], particularly with dissipative dynamics. Indeed, the "unwinding," or annihilation, of textures with antitextures appears not to have been addressed in two dimensions. In this paper we consider mixed systems of textures and antitextures developed by instantaneously quenching a disordered state to $T=0$. The phase-ordering dynamics are taken to be local, purely dissipative dynamics. For these systems, our numerical simulations find a rich pattern of scaling violations.

The nonconserved "model A" dynamics are

$$\partial_t \vec{\phi}(\mathbf{x},t) = -\delta H / \delta \vec{\phi}, \tag{1}$$

where

$$H = \int d^2x [(\nabla \vec{\phi})^2 + V_0(\vec{\phi}^2 - 1)^2]. \tag{2}$$

The lack of thermal noise in (1) is appropriate for a quench to $T=0$, where the equilibrium state has long-range order. We simulate the dynamics numerically with both soft ($V_0<\infty$) and hard ($V_0=\infty$, or equivalently with a $|\vec{\phi}|=1$ constraint) spins. We consider systems on square lattices of sizes between 128×128 and 512×512, with periodic boundary conditions and independent randomly oriented unit-magnitude spins as initial conditions. We use at least 20 independent runs for each system, and errors indicated in the figures are extracted from the variations between runs. We use a simple Euler update with a fixed time scale $\delta t=0.01$ (except for $V_0=1/2$, where we used $\delta t=0.1$), and consider times up to $t=10\,000$. The late-time regime covers nearly three decades of time, starting with times $t\gtrsim10$. This regime is unchanged with a smaller time step. Similar results are obtained with different V_0, although the early-time behavior, and hence the asymptotic texture density, changes if V_0 is small enough. In all cases, however, the asymptotic growth laws remain the same.

Previous numerical work has been carried out on these 2D O(3) systems, using Eqs. (1) and (2), with both hard spins

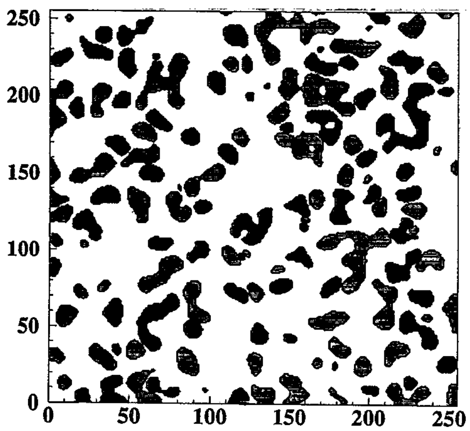

FIG. 1. A snapshot of one-quarter of a 512×512 system with $V_0=10$, at $t=13.1$. Shown are the regions with texture density above the average magnitude, $|\rho|\geqslant\langle|\rho|\rangle$. Black clusters have positive texture density, while outlined gray clusters have negative texture density.

[11] and with soft spins ($V_0=10$) [12], although topological quantities were not measured. The hard-spin simulations, by Bray and Humayun, found an energy density consistent with $\epsilon\sim t^{-2/3}$, indicating a length scale $L\sim t^{1/3}$, and no dynamical scaling [11]. The soft-spin simulations, by Toyoki, found $L\sim t^{0.42\pm0.03}$ and dynamical scaling; however, only the energy and spin-spin correlations at times $t\lesssim20$ (in our units) were investigated [12]. In this regime, we see strong early-time transients. Our results are consistent with this previous work. However, we identify a distinct late-time regime in which scaling is violated, and with growth law exponents that do not depend on V_0.

The topological texture density for these systems is given by

$$\rho(\mathbf{x})=[\vec{\phi}\cdot(\partial_x\vec{\phi}\times\partial_y\vec{\phi})]/4\pi, \tag{3}$$

which measures the local winding of the spin field around the unit sphere in order-parameter space. We show two snapshots of the texture density for part of a single 512×512 system in Figs. 1 and 2. For clarity, we only show texture density above the average magnitude in each figure. We identify the three evolving length scales: L_T characterizes the scale of each texture, L_N characterizes the separation between textures independent of charge, and L_C characterizes the separation between oppositely charged textures.

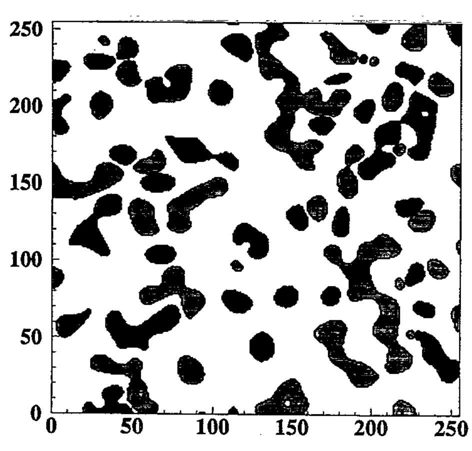

FIG. 2. A snapshot of the same system and region as in the last figure, at $t=27.9$. Again, we show $|\rho|\geqslant\langle|\rho|\rangle$.

Dimensionally the equation of motion (1) determines a growing length scale $L\sim t^{1/2}$, where we have suppressed the dimensioned kinetic coefficient. Growth laws different from this are only possible through the introduction of other lengths: either the core scale $\xi\sim V_0^{-1/2}$, or the initial correlation length ξ_0. We do not expect any ξ dependence in this system since there are no singular defects. Any ξ_0 dependence, on the other hand, indicates a scaling violation, because then changing ξ_0 is not just equivalent to a shift in time [4]. So, for systems without singular defects, any growth law that differs from the dimensionally naive one indicates a scaling violation. In addition, only positive powers of ξ_0 can enter into any growth law, yielding smaller growth exponents than 1/2, because shorter initial correlations should only decrease asymptotic length scales.

We first show the energy density and the average magnitude of the texture density ρ in Fig. 3(a). The energy density ϵ follows from Eq. (2), and dimensionally scales as an inverse length-squared. The texture density, from Eq. (3), has the same scaling. For spins of unit magnitude in the continuum, $\langle\rho\rangle$ is a conserved quantity. However, $\langle|\rho|\rangle$ is not, and it decreases along with ϵ as textures unwind with antitextures. At late times ϵ and $\langle|\rho|\rangle$ fall off as $t^{-0.65\pm0.02}$, indicating a growing length scale $L_N\sim t^{0.33\pm0.01}$. (We fit times with $t>10$ and extract the approximate errors from the variance of the exponents between different system sizes and different V_0.) This length scale, L_N, determines the overall texture density, and so characterizes the separation of individual textures. There is excellent quantitative agreement between the different system sizes, indicating that finite size effects only occur at later times, and qualitative agreement between the soft- and hard-spin simulations at late times. For soft potentials, the spins are not saturated at early times and this results in the initial increase of the average magnitude of the texture density seen for $V_0=1/2$, and the shift in the asymptotic texture density at late times.

We can also consider the portion of the energy contained in textures. Since an isolated hard-spin texture has an energy of 8π using (2) [9], we have plotted $8\pi\langle|\rho|\rangle$ in Fig. 3(a). The agreement with ϵ at late times indicates that the asymptotic energy density is wholly contained in the textures. The difference, $\epsilon-\langle|\rho|\rangle$ (not shown), decays very roughly like t^{-1}—and indicates a nontexture early-time transient. This could explain why the strong early-time correction to scaling observed in ϵ is much smaller in the texture density $\langle|\rho|\rangle$.

While we find no way to scale two-point correlations of either spins or texture density, we can extract relevant length scales from the evolution of the correlations. However, because of the lack of scaling, this analysis has significant systematic errors due to the evolving functional form of the correlations. In Fig. 3(b), we show the position of the first zero of

$$S(r,t)\equiv\langle\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})\rangle. \tag{4}$$

This zero roughly characterizes the texture-antitexture sepa-

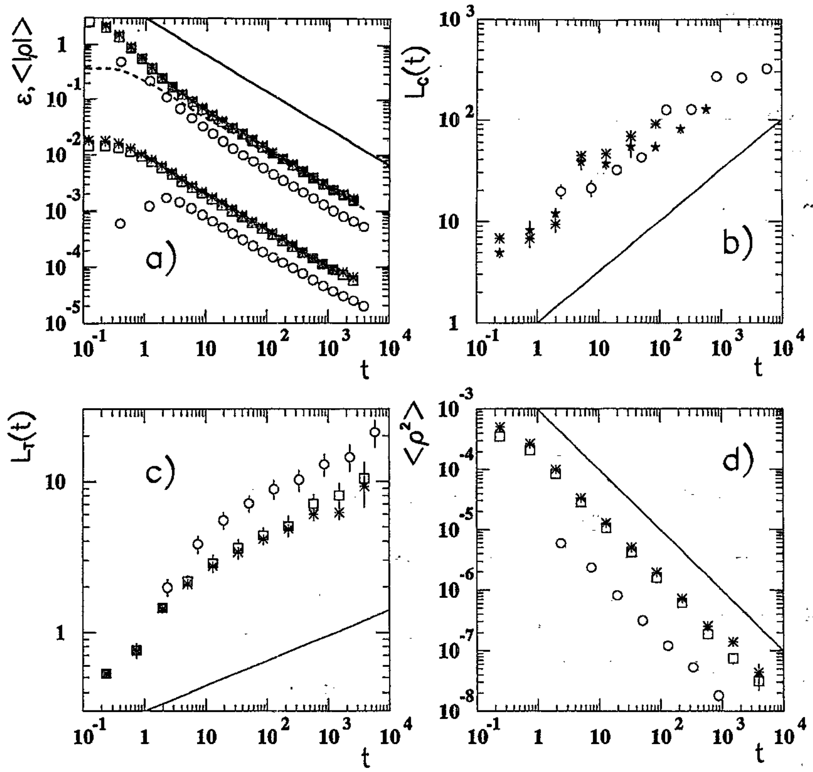

FIG. 3. (a) The energy density ϵ (top plots) and the texture density $\langle|\rho|\rangle$ (bottom plots) vs time after the quench. The circles indicate a system size $L_\infty=512$ with $V_0=1/2$, the squares indicate $L_\infty=512$ with $V_0=10$, and the asterisks indicate $L_\infty=128$ with $V_0=\infty$. The straight line has slope $-2/3$. The dashed line is $8\pi\langle|\rho|\rangle$ from the $L_\infty=512$ system with $V_0=10$. (b) The texture-antitexture separation L_C, characterized by the position of the first zero of $S(r,t)$ vs time after the quench. The symbols are the same, with stars indicating $L_\infty=256$ with $V_0=10$. The straight line has slope 1/2. (c) The texture size L_T, characterized by the width, at half-height, of the first peak of $A(r,t)$ vs time since the quench. The straight line has slope 1/6. (d) The average square texture density vs time since the quench. The straight line has slope -1.

ration, L_C. The growth saturates as L_C approaches the system size. We find $L_C\sim t^{0.4\pm0.1}$, fit over times $t>10$ and excluding late times when finite size effects are apparent.

There is a heuristic energy-scaling argument [4] for $L_C\sim t^{1/2}$. We first identify the energy density of the evolving system, $\epsilon\sim 1/L_N^2$. Independently, we calculate the rate of energy-density dissipation, $\partial_t\epsilon=\langle\partial_t\vec{\phi}\;\delta H/\delta\vec{\phi}\rangle=-\langle(\partial_t\vec{\phi})^2\rangle$, where we have used (1). Applying the dynamics to (2), and neglecting the potential term since it is subdominant energetically [4], we find $\partial_t\vec{\phi}\simeq\nabla^2\vec{\phi}$. Inside a texture, $\nabla\vec{\phi}\sim 1/L_T$, while the Laplacian vanishes identically for pure texture solutions [9]. The natural length to enter in the second derivative, then, is the texture-antitexture separation, L_C, so we expect $\nabla^2\vec{\phi}\sim 1/L_TL_C$. This will hold over the area of the texture (L_T^2) for each texture (one per L_N^2), resulting in $\partial_t\epsilon\sim 1/(L_TL_C)^2L_T^2/L_N^2\sim 1/L_C^2L_N^2$. Comparing with $\epsilon\sim 1/L_N^2$, we find $L_C\sim t^{1/2}$. This growth law is consistent with the data.

What of the third length scale, L_T? Since L_T is the texture scale, it characterizes the width of the peak centered at $r=0$ of the two-point texture-density correlations. In Fig. 3(c), we have plotted the width at half-maximum of

$$A(r,t)\equiv\langle|\rho(\mathbf{x})\rho(\mathbf{x}+\mathbf{r})|\rangle-\langle|\rho|\rangle^2. \qquad (5)$$

Fitting the points for $t>10$ we find $L_T\sim t^{0.21\pm0.02}$, with similar results from $S(r,t)$. We can check this growth law by measuring $\langle\rho^2\rangle$. Since there is one texture, of area L_T^2 and with $\rho^2\sim 1/L_T^4$, in each area L_N^2, we expect $\langle\rho^2\rangle\sim 1/(L_N^2L_T^2)$. In Fig. 3(d), we measure $\langle\rho^2\rangle\sim t^{-1.02\pm0.03}$, by fitting $t>10$. Using $L_N^2\sim t^{0.65\pm0.02}$ from ϵ, we have an independent estimate of $L_T\sim t^{0.19\pm0.02}$. This is consistent, and we combine these estimates to find $L_T\sim t^{0.20\pm0.02}$.

From the heuristic picture presented for L_C, small textures have larger gradients and will annihilate earlier — leading to an increasing L_T. If we assume that the remaining textures have not evolved, then their internal charge density, $1/L_T^2$, is set by the initial fluctuations on the scale L_N, of order L_N/L_N^2. This implies $L_T\sim L_N^{1/2}$. Using our measurements for L_T and L_N, it is consistent that $L_N\sim t^{1/3}$ and $L_T\sim t^{1/6}$ (where $\langle\rho^2\rangle\sim t^{-1}$). These exponents have been indicated by straight lines in Fig. 3.

To summarize, we have found three characteristic length scales in the phase ordering of the 2D nonconserved Heisenberg model. These length scales exhibit different growth laws and so demonstrate the violation of dynamical scaling in this system. The growth laws are $L_C\sim t^{0.4\pm0.1}$ describing the separation of textures and antitextures, $L_N\sim t^{0.33\pm0.01}$ characterizing the separation of textures, and $L_T\sim t^{0.20\pm0.02}$ describing the scale of individual textures. From our heuristic arguments, we believe that $L_C\sim t^{1/2}$ exactly, and our data are consistent with $L_N\sim t^{1/3}\xi_0^{1/3}$ and $L_T\sim t^{1/6}\xi_0^{2/3}$. The factors of the initial correlation length ξ_0 make up the dimensions of length needed for the scaling violations. A more convincing explanation for the growth laws awaits a detailed understanding of the texture-antitexture unwinding mechanism.

Soft- and hard-spin simulations give the same growth laws. Individual textures are destabilized by a finite V_0 which leads to an effective negative $(\nabla\vec{\phi})^4$ contribution to the local energy density [6]. Scaling and numerical treatments of isolated textures, with $V_0<\infty$, then give a texture scale $X(\tilde{t})\sim\tilde{t}^{1/4}$, where $\tilde{t}$ is the time remaining to collapse. The near-neighbor lattice gradient used had similar contributions and instabilities [13]. The slightly faster than expected growth of L_T may come from the the resulting collapse of small textures. However, clarifying the role of the weak texture instabilities in the asymptotic late-time regime awaits a study with arbitrary, rather than just negative, fourth-order gradients in the lattice version of (2).

To what extent does our picture of scaling violations carry over to other systems with topological textures? Certainly in one dimension a similar picture holds [5]. We believe that the observed scaling violations in 1D and 2D are due to the weak interactions between textures. This being the case, we would expect similar scaling violations in conserved 2D O(3) models, though these have not yet been explored to our knowledge. However, because individual textures are strongly unstable for $d>2$ [14], it would be surprising if similar scaling violations were observed in higher dimensions.

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